

## 2011 Paper 7 Question 5

### Denotational Semantics

(a) Let  $\Omega$  be the domain

$$0 \sqsubseteq 1 \sqsubseteq \dots \sqsubseteq n \sqsubseteq \dots \sqsubseteq \omega \quad (n \in \mathbb{N})$$

(That is,  $\Omega = (\mathbb{N} \cup \{\omega\}, \sqsubseteq)$  with  $x \sqsubseteq y$  in  $\Omega$  iff  $x \leq y$  in  $\mathbb{N}$  or  $y = \omega$ .)

Indicate whether the following statements are true or false. Provide an argument for each answer.

(i) Every monotone function from  $\Omega$  to  $\Omega$  is continuous. [5 marks]

(ii) Every monotone function from  $\Omega$  to  $\Omega$  has a least pre-fixed point. [5 marks]

(b) Let  $D$  and  $E$  be domains, and let  $f : D \rightarrow D$  and  $g : E \rightarrow E$  be continuous functions.

(i) Define  $f \times g : D \times E \rightarrow D \times E$  to be the continuous function given by  $(f \times g)(d, e) = (f(d), g(e))$ , and let  $\pi_1 : D \times E \rightarrow D$  and  $\pi_2 : D \times E \rightarrow E$  respectively denote the first and second projection functions.

Show that  $fix(f \times g) \sqsubseteq (fix(f), fix(g))$ , and that  $fix(f) \sqsubseteq \pi_1(fix(f \times g))$  and  $fix(g) \sqsubseteq \pi_2(fix(f \times g))$ . [5 marks]

(ii) It follows from part (i) that  $fix(f \times g) = (fix(f), fix(g))$ . Use this and Scott's Fixed Point Induction Principle to show that, for all strict continuous functions  $h : D \rightarrow E$ ,

$$h \circ f = g \circ h \implies h(fix(f)) = fix(g) \quad [5 \text{ marks}]$$