Types

Consider the following type and expressions of the Polymorphic Lambda Calculus (PLC):

\[ nat = \forall \alpha (\alpha \to (\alpha \to \alpha)) \]
\[ Z = \Lambda \alpha (\lambda x : \alpha (\lambda f : \alpha \to \alpha (x))) \]
\[ S = \lambda y : nat (\Lambda \alpha (\lambda x : \alpha (\lambda f : \alpha \to \alpha (f (y \alpha x f)))))) \]

(a) What are the types of \(Z\) and \(S\)? [2 marks]

(b) Show that there is a closed PLC expression \(I\) of type

\[ \forall \alpha (\alpha \to (\alpha \to \alpha)) \to \text{nat} \to \alpha \]

satisfying the following beta-conversions:

\[ I \alpha x f Z =_\beta x \]
\[ I \alpha x f (S y) =_\beta f (I \alpha x f y) \]

[4 marks]

(c) For each natural number \(n \in \mathbb{N} = \{0, 1, 2, \ldots\}\), let \(S^n Z\) be the PLC expression given by

\[ S^0 Z = Z \]
\[ S^{n+1} Z = S(S^n Z) \]

What is the beta-normal form of \(S^0 Z\), of \(S^1 Z\), of \(S^2 Z\), and in general, of \(S^n Z\)? [4 marks]

(d) (i) Using part (b), or otherwise, show that there is a closed PLC expression \(A\) of type \(\text{nat} \to \text{nat} \to \text{nat}\) that represents addition of natural numbers, in the sense that \(A(S^m Z)(S^n Z) =_\beta S^{m+n} Z\) holds for all \(m, n \in \mathbb{N}\). [Hint: recall the primitive recursive definition of addition.] [5 marks]

(ii) Show that \(M = \lambda y : \text{nat} (I \text{n}at Z (A y))\) (with \(A\) as in part (i)) represents multiplication of natural numbers, in the sense that \(M(S^m Z)(S^n Z) =_\beta S^{mn} Z\) holds for all \(m, n \in \mathbb{N}\). [5 marks]