Computation Theory

(a) Define Church’s representation of numbers \( n \) as \( \lambda \)-terms \( n \). [3 marks]

(b) What does it mean for a partial function \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) to be \( \lambda \)-definable? What is the relationship between \( \lambda \)-definability and computability? [3 marks]

(c) Show that \( \text{succ}(x_1) = x_1 + 1 \) is \( \lambda \)-definable. [4 marks]

(d) Ackermann’s function \( \text{ack} \in \mathbb{N}^2 \rightarrow \mathbb{N} \) is a total function of two arguments satisfying

\[
\begin{align*}
\text{ack}(0, x_2) &= x_2 + 1 \\
\text{ack}(x_1 + 1, 0) &= \text{ack}(x_1, 1) \\
\text{ack}(x_1 + 1, x_2 + 1) &= \text{ack}(x_1, \text{ack}(x_1 + 1, x_2)).
\end{align*}
\]

By considering \( \lambda x. x T S \) where \( T = \lambda f y. y f (f 1) \) and \( S \) is chosen suitably, prove that Ackermann’s function is \( \lambda \)-definable. [10 marks]