Computer Systems Modelling

(a) Suppose that $X$ is a random variable having the Binomial distribution with parameters $n$ and $p$ and that $\lambda > 0$ is a constant.

(i) Write down the expression for $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, \ldots, n\}$.

(ii) Now suppose that $n \to \infty$ and $p$ is chosen so that $p = \lambda/n$. Show that under this limit $\mathbb{P}(X = k) \to e^{-\lambda} \lambda^k / k!$, that is, to a Poisson distribution with parameter $\lambda$.

(b) Suppose that $N(t)$ is the random number of events in the time interval $[0, t]$ of a Poisson process with parameter $\lambda$.

(i) State the conditions that define the Poisson process $N(t)$.

(ii) Show that for all $t > 0$ the random variable $N(t)$ has the Poisson distribution with parameter $\lambda t$.

(c) Given a Poisson process of rate $\lambda$ let $X_1$ be the time of the first event and for $n > 1$ let $X_n$ denote the time between the events $(n - 1)$ and $n$. Thus the sequence $X_1, X_2, \ldots$ gives us the sequence of inter-event times between the events in a Poisson process.

(i) Show that $\mathbb{P}(X_1 > t) = \mathbb{P}(N(t) = 0)$ for $t > 0$.

(ii) Show that the inter-event times $X_1, X_2, \ldots$ are independent, identically distributed random variables each of whose marginal distribution is an Exponential with rate parameter $\lambda$. 