2009 Paper 2 Question 6

Discrete Mathematics II

(a) A partial order $(P, \leq)$ comprises a set P together with a binary relation $\leq$ which is reflexive, transitive and antisymmetric. Explain what the terms *reflexive*, *transitive* and *antisymmetric* mean. [3 marks]

(b) The relation $\leq$ on natural numbers $\mathbb{N} = \{1,\ 2,\ \cdots\}$ is defined by

$$m \leq n \text{ iff } m \text{ divides } n,\ \text{ that is } m \cdot k = n \text{ for some integer } k\,.$$

Invoking standard facts about division, establish that $\leq$ is a partial order. If in the definition of $\leq$ we used the set of all integers $\mathbb{Z}$, instead of $\mathbb{N}$, would $(\mathbb{Z}, \leq)$ be a partial order? Explain your answer briefly. [5 marks]

(c) Draw the Hasse diagram for $\leq$ on the set $\{1, 2, \cdots, 13\}$. Identify the greatest lower bound (glb) and least upper bound (lub) of $\{4,\ 6\}$. Does the partial order $(\mathbb{N}, \leq)$ have greatest lower bounds and least upper bounds of all subsets of $\mathbb{N}$, including all infinite subsets? Explain your answers briefly. [6 marks]

(d) An *atom* of the partial order $(\mathbb{N}, \leq)$ is an element $a \in \mathbb{N}$ such that

$$\forall x \in \mathbb{N}.\ (1 \leq x \text{ and } x \leq a) \Rightarrow (1 = x \text{ or } x = a)\,.$$

Identify the atoms in your Hasse diagram, and more generally in $\mathbb{N}$. [3 marks]

(e) Explain, without proof, why a partial order that has least upper bounds of all subsets also has greatest lower bounds of all subsets. [3 marks]