Denotational Semantics

(a) Describe the properties a function between two cpos must have to be continuous. [2 marks]

(b) Let $D_1$, $D_2$ and $E$ be cpos. Prove that a function $h : D_1 \times D_2 \rightarrow E$ is continuous if it is continuous in each argument separately. [You may assume standard properties of least upper bounds provided you state them clearly.] [4 marks]

(c) Let $\mathbb{O}$ be the cpo with two elements $\bot \subseteq \top$. For a cpo $E$ and $e \in E$, define the function $g_e : E \rightarrow \mathbb{O}$ by

$$g_e(x) = \begin{cases} \bot & \text{if } x \sqsubseteq e \\ \top & \text{if } x \nsubseteq e \end{cases}$$

Show $g_e$ is continuous. [4 marks]

(d) As an example of the definition in part (c) above, let $E = B_{\bot} \times B_{\bot}$, where $B = \{true, false\}$, and consider $g_{(false,false)} : E \rightarrow \mathbb{O}$. Show that

$$g_{(false,false)}(x, y) = \top \text{ iff } x = true \text{ or } y = true$$

[2 marks]

(e) Let $f : D \rightarrow E$ be a function between cpos $D$ and $E$. Show

$$f \text{ is continuous } \text{ iff } \forall e \in E. \ g_e \circ f \text{ is continuous}$$

[You may assume that the composition of continuous functions is continuous. It is suggested that for the “if” direction of the proof, you argue by contradiction.] [8 marks]