Mathematical Methods for Computer Science

(a) Consider a *simple random walk*, $S_n$, defined by $S_0 = a$ and $S_n = S_{n-1} + X_n$ for $n \geq 1$ where the random variables $X_i$ ($i = 1, 2, \ldots$) are independent and identically distributed with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ for some constant $p$ with $0 \leq p \leq 1$.

(i) Find $E(S_n)$ and $Var(S_n)$ in terms of $a$, $n$ and $p$. [4 marks]

(ii) Use the *central limit theorem* to derive an approximate expression for $P(S_n > k)$ for large $n$. You may leave your answer expressed in terms of the distribution function $\Phi(x) = P(Z \leq x)$ where $Z$ is a standard Normal random variable with zero mean and unit variance. [6 marks]

(b) Consider the *Gambler’s ruin problem* defined as in part (a) but with the addition of absorbing barriers at 0 and $N$ where $N$ is some positive integer. Derive an expression for the probability of ruin (that is, being absorbed at the zero barrier) when starting at position $S_0 = a$ for each $a = 0, 1, \ldots, N$ in the two cases

(i) $p \neq \frac{1}{2}$ [5 marks]

(ii) $p = \frac{1}{2}$. [5 marks]