Denotational Semantics

(a) Suppose that \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\) are cpos.

\( (i) \) What properties does a function \( f : D \to E \) need to satisfy in order to be continuous? \hspace{1cm} [2 marks]

\( (ii) \) Assume also that \((C, \sqsubseteq_C)\) is a cpo and that \( g : C \times D \to E \) is a continuous function. Let \( g^* : C \to (D \to E) \) be defined by \( g^*(c) = \lambda d \in D. g(c,d) \). Prove that \( g^* \) is continuous. You may refer to general facts about least upper bounds in product and function cpos provided that you state them clearly. \hspace{1cm} [6 marks]

(b) Let \( 2 = (\{\bot, \top\}, \bot \sqsubseteq \top) \) be the unique domain with two elements.

\( (i) \) Draw a diagram which represents the elements of the function domain \( 2 \to 2 \) and shows their ordering; \hspace{1cm} [1 mark]

\( (ii) \) Any set \( X \) can be considered as a flat domain \( X_\bot \) by adding a bottom element. Show that the strict continuous functions \( X_\bot \to 2 \) are in 1-1 correspondence with the subsets of \( X \). \hspace{1cm} [2 marks]

(c) Define what is meant by an admissible subset of a domain \( D \). \hspace{1cm} [2 marks]

(d) State the principle of Scott induction and prove its validity. \hspace{1cm} [4 marks]

(e) Suppose that \( D \) is a domain and \( f : D \times D \to D \) is a continuous function satisfying the property \( \forall d, e \in D. f(d,e) = f(e,d) \). Let \( g : D \times D \to D \times D \) be defined by \( g(d_1, d_2) = (f(d_1, f(d_1, d_2)), f(f(d_1, d_2), d_2)) \). Let \( (u_1, u_2) = \text{fix}(g) \). Show that \( u_1 = u_2 \) using Scott induction. \hspace{1cm} [3 marks]