Mathematics for Computation Theory

State the requirements for \((S, \leq)\) to be:

(a) a partially ordered set;

(b) a totally ordered set;

(c) a well ordered set.  

Let \(\mathbb{N}\) be the natural numbers. Give, without proof, three examples of relations \(\leq_i\), where \(i = 1, 2, 3\), such that \((\mathbb{N}, \leq_i)\) satisfies exactly \(i\) of the conditions (a), (b), (c).

Let \(S = \{a, b\}\) be an alphabet, with total order \(a < b\). Let \(\Sigma = S^*\) be the set of all strings over \(S\); for \(w = s_1 s_2 \ldots s_n \in \Sigma\) we write \(\ell(w) = n\), and for \(1 \leq r \leq n = \ell(w)\) we write \(w_r = s_1 s_2 \ldots s_r\). Denote by \(\varepsilon\) the unique word of \(\Sigma\) such that \(\ell(\varepsilon) = 0\), the null string. Conventionally \(w_0 = \varepsilon\) for all words \(w \in \Sigma\).

Define relation \(<\) on \(\Sigma\) as follows:

Let \(v, w \in \Sigma\), and \(n = \min\{\ell(v), \ell(w)\}\). Let \(r = \max\{i \mid v_i = w_i\}\) \(\leq n\).

Then \(v < w\) if:

either \(\begin{align*}
(i) & \quad \ell(v) = r; \\
(ii) & \quad v_r+1 = v_r a, \quad w_r+1 = w_r b, \quad \text{where } v_r = w_r.
\end{align*}\)

Which of conditions (a), (b), (c) above are satisfied by \((\Sigma, <)\)?