Types

Given any polymorphic lambda calculus (PLC) type $\tau$ and any function $\rho$ mapping type variables $\alpha$ to values $n \in \{-1, 0, 1\}$, a value $\llbracket \tau \rrbracket \rho$ in $\{-1, 0, 1\}$ is defined by recursion on the structure of $\tau$ as follows.

$$\llbracket \alpha \rrbracket \rho = \rho(\alpha)$$

$$\llbracket \tau_1 \to \tau_2 \rrbracket \rho = \begin{cases} 1 & \text{if } \llbracket \tau_1 \rrbracket \rho \leq \llbracket \tau_2 \rrbracket \rho \\ \llbracket \tau_2 \rrbracket \rho & \text{otherwise} \end{cases}$$

$$\llbracket \forall \alpha(\tau) \rrbracket \rho = \text{the minimum of the values } \llbracket \tau \rrbracket(\rho[\alpha \mapsto n]) \text{ for } n = -1, 0, 1 \text{ (where } \rho[\alpha \mapsto n] \text{ is the function mapping } \alpha \text{ to } n \text{ and every other } \alpha' \text{ to } \rho(\alpha')).$$

If $\Gamma$ is a non-empty PLC typing environment, let $\llbracket \Gamma \rrbracket \rho$ denote the minimum value of $\llbracket \tau \rrbracket \rho$ as $\tau$ ranges over the types in $\Gamma$; in the case that $\Gamma$ is empty, we define $\llbracket \Gamma \rrbracket \rho$ to be 1.

(a) Prove that if $\Gamma \vdash M : \tau$ is a valid PLC typing judgement, then for any $\rho$, $\llbracket \Gamma \rrbracket \rho \leq \llbracket \tau \rrbracket \rho$.

You may assume without proof that if $\alpha$ is not free in $\tau$ then

$$\llbracket \tau \rrbracket(\rho[\alpha \mapsto n]) = \llbracket \tau \rrbracket \rho$$

and also that type substitutions $\tau[\tau'/\alpha]$ satisfy

$$\llbracket \tau[\tau'/\alpha] \rrbracket \rho = \llbracket \tau \rrbracket(\rho[\alpha \mapsto \llbracket \tau' \rrbracket \rho])$$

[Hint: show that the property $\Phi(\Gamma, M, \tau) = \text{"for all } \rho, \llbracket \Gamma \rrbracket \rho \leq \llbracket \tau \rrbracket \rho \text{" is closed under the rules of the typing system.}]

[16 marks]

(b) Deduce that there is no closed PLC expression of type

$$\forall \alpha, \beta(((\alpha \to \beta) \to \alpha) \to \alpha)$$

[4 marks]