2006 Paper 2 Question 5

Discrete Mathematics II

(a) What does it mean for a function to be an *injection*, *surjection* and *bijection*? [3 marks]

(b) Let $I = \{x \in \mathbb{R} \mid x > 1\}$. Define a binary relation $g \subseteq I \times I$ by taking

$$(u, v) \in g \text{ iff } \frac{1}{u} + \frac{1}{v} = 1 .$$

(i) Express v as a formula in u for $(u, v) \in g$. Deduce that g is a function $g : I \to I$. [2 marks]

(ii) State what properties are required of a function $h : I \to I$ in order for h to be an inverse function to g. Define an inverse function to g and prove that it has the desired properties. Deduce that $g : I \to I$ is a bijection. [6 marks]

(c) (i) Let X be a set. Prove there is no injection $f : \mathcal{P}(X) \to X$.
[Hint: consider the set $Y \stackrel{\text{def}}{=} \{f(Z) \mid Z \subseteq X \ \wedge f(Z) \notin Z\}$.] [5 marks]

(ii) Suppose now that the set X has at least two distinct elements. Define an injection $k : \mathcal{P}(X) \to (X \to X)$, from the powerset of X to the set of functions from X to X. [2 marks]

(iii) Prove that there is no injection from $(X \to X)$ to X when the set X has at least two distinct elements. [You may assume that the composition of injections is an injection.] [2 marks]