Numerical Analysis II

(a) In Peano’s theorem, if a quadrature rule integrates polynomials of degree \( N \) exactly over an interval \([a, b]\), then the error in integrating \( f \in C^{N+1}[a, b] \) is expressed as

\[
E(f) = \int_a^b f^{(N+1)}(t)K(t) \, dt
\]

where

\[
K(t) = \frac{1}{N!} E_x[(x - t)^N].
\]

Explain the notation \( E(f), E_x, (x - t)^N \). [4 marks]

(b) Assuming \( x \in [a, b] \), and writing Taylor’s theorem in the form

\[
f(x) = P_N(x - a) + \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x - t)^N \, dt
\]

where \( P_N \) is a polynomial of degree \( N \), prove Peano’s theorem, explaining each step clearly. [8 marks]

(c) For the trapezium rule, what is \( N \)? [1 mark]

(d) If \( K(t) \) does not change sign in \([a, b]\) then

\[
E(f) = \frac{f^{(N+1)}(\xi)}{(N+1)!} E(x^{N+1})
\]

for some \( \xi \in (a, b) \). Use this result to simplify

\[
E(f) = \int_{-1}^1 f(x) \, dx - f(-1) - f(1).
\]

[7 marks]