Numerical Analysis I

(a) The Newton–Raphson iteration for solution of \( f(x) = 0 \) is

\[
\tilde{x} = x - \frac{f(x)}{f'(x)}. 
\]

By drawing a carefully labelled graph, explain the graphical interpretation of this formula. What is the order of convergence? [4 marks]

(b) Consider \( f(x) = x^3 + x^2 - 2 \). The following table shows successive iterations for each of the three starting values (i) \( x = 1.5 \), (ii) \( x = 0.2 \), (iii) \( x = -0.5 \). Note that, to the accuracy shown, each iteration finds the root at \( x = 1 \).

\[
\begin{array}{cccc}
 n & (i) & (ii) & (iii) \\
 0 & 1.50000 \times 10^0 & 2.00000 \times 10^{-1} & -5.00000 \times 10^{-1} \\
 1 & 1.12821 \times 10^0 & 3.95384 \times 10^0 & -8.00000 \times 10^0 \\
 2 & 1.01152 \times 10^0 & 2.57730 \times 10^0 & -5.44318 \times 10^0 \\
 3 & 1.00010 \times 10^0 & 1.70966 \times 10^0 & -3.72976 \times 10^0 \\
 4 & 1.00000 \times 10^0 & 1.22393 \times 10^0 & -2.56345 \times 10^0 \\
 5 & 1.00000 \times 10^0 & 1.03212 \times 10^0 & -1.72202 \times 10^0 \\
 6 & 1.00079 \times 10^0 & 1.00079 \times 10^0 & -9.62478 \times 10^{-1} \\
 7 & 1.00000 \times 10^0 & 1.00000 \times 10^0 & 1.33836 \times 10^0 \\
 8 & 1.00000 \times 10^0 & 1.00000 \times 10^0 & 1.06651 \times 10^0 \\
 9 & 1.00329 \times 10^0 & 1.00000 \times 10^0 & 1.00000 \times 10^0 \\
 10 & 1.00000 \times 10^0 & 1.00000 \times 10^0 & 1.00000 \times 10^0 \\
 11 & 1.00000 \times 10^0 & 1.00000 \times 10^0 & \\
\end{array}
\]

Sketch the graph of \( f(x) \) and show the first iteration for cases (i) and (ii) to show why (i) converges faster than (ii). In a separate sketch, show the first two iterations for case (iii). [Hint: a very rough sketch will suffice for case (iii).] [10 marks]

(c) Now consider \( f(x) = x^4 - 3x^2 - 2 \). Calculate two Newton–Raphson iterations from the starting value \( x = 1 \). Comment on the prospects for convergence in this case. [6 marks]