Denotational Semantics

Let $D$ be a domain with bottom element $\bot$. Let $h, k : D \rightarrow D$ be continuous functions with $h$ strict (so $h(\bot) = \bot$). Let $\mathbb{B} = \{true, false\}$. Define the conditional function,

$$if : \mathbb{B}_\bot \times D \times D \rightarrow D$$

by $if(b, d, d') = d$ if $b = true$, $d'$ if $b = false$, and $\bot$ otherwise. Let $p : D \rightarrow \mathbb{B}_\bot$ be a continuous function.

The function $f$ is the least continuous function from $D \times D$ to $D$ such that

$$\forall x \in D. f(x, y) = if(p(x), y, h(f(k(x), y))) .$$

(a) State the principle of fixed point induction. What does it mean for a property to be admissible? [4 marks]

(b) Show that

$$\forall b \in \mathbb{B}_\bot, d, d' \in D. h(if(b, d, d')) = if(b, h(d), h(d')) .$$

[3 marks]

(c) Prove that the property

$$Q(g) \iff \forall x, y \in D. h(g(x, y)) = g(x, h(y)) ,$$

where $g$ is a continuous function from $D \times D$ to $D$, is admissible. [5 marks]

(d) Prove $Q(f)$ by fixed point induction. [8 marks]