Information Theory and Coding

(a) For continuous random variables $X$ and $Y$, taking on continuous values $x$ and $y$ respectively with probability densities $p(x)$ and $p(y)$ and with joint probability distribution $p(x,y)$ and conditional probability distribution $p(x|y)$, define:

(i) the differential entropy $h(X)$ of random variable $X$; [1 mark]

(ii) the joint entropy $h(X,Y)$ of the random variables $X$ and $Y$; [1 mark]

(iii) the conditional entropy $h(X|Y)$ of $X$, given $Y$; [1 mark]

(iv) the mutual information $i(X;Y)$ between the continuous random variables $X$ and $Y$; [1 mark]

(v) how the channel capacity of a continuous channel which takes $X$ as its input and emits $Y$ as its output would be determined. [1 mark]

(b) For a time-varying continuous signal $g(t)$ which has Fourier transform $G(k)$, state the modulation theorem and explain its rôle in AM radio broadcasting. How does modulation enable many independent signals to be encoded into a common medium for transmission, and then separated out again via tuners upon reception? [4 marks]

(c) Briefly define

(i) The Differentiation Theorem of Fourier analysis: if a function $g(x)$ has Fourier transform $G(k)$, then what is the Fourier transform of the $n^{th}$ derivative of $g(x)$, denoted $g^{(n)}(x)$? [2 marks]

(ii) If discrete symbols from an alphabet $S$ having entropy $H(S)$ are encoded into blocks of length $n$, we derive a new alphabet of symbol blocks $S^n$. If the occurrence of symbols is independent, then what is the entropy $H(S^n)$ of the new alphabet of symbol blocks? [2 marks]

(iii) If symbols from an alphabet of entropy $H$ are encoded with a code rate of $R$ bits per symbol, what is the efficiency $\eta$ of this coding? [2 marks]

(d) Briefly explain

(i) how $10 \text{ V}$ is expressed in dB$\mu$V; [1 mark]

(ii) the YCrCb coordinate system. [4 marks]