Computation Theory

(a) What does it mean for a subset \( S \) of the set \( \mathbb{N} \) of natural numbers to be register machine \textit{decidable}? [3 marks]

(b) For each \( e \in \mathbb{N} \), let \( \varphi_e \in Pfn(\mathbb{N}, \mathbb{N}) \) denote the partial function computed by the register machine program with index \( e \). Let \( e_0 \in \mathbb{N} \) be an index for the totally undefined partial function (so that \( \varphi_{e_0}(x) \uparrow \), for all \( x \in \mathbb{N} \)).

Suppose that a total function \( f \in Fun(\mathbb{N}, \mathbb{N}) \) is \textit{extensional}, in the sense that for all \( e, e' \in \mathbb{N} \), \( f(e) = f(e') \) if \( \varphi_e \) and \( \varphi_{e'} \) are equal partial functions. Suppose also that the set \( S_f = \{ x \in \mathbb{N} \mid f(x) = f(e_0) \} \) is not the whole of \( \mathbb{N} \), so that for some \( e_1 \in \mathbb{N} \), \( f(e_1) \neq f(e_0) \).

(i) If membership of \( S_f \) were decided by a register machine \( M \), show informally how to construct from \( M \) a register machine \( M' \) that, started with \( R_1 = e \) and \( R_2 = n \) (any \( e, n \in \mathbb{N} \)) always halts, with \( R_0 = 0 \) if \( \varphi_e(n) \downarrow \) and with \( R_0 = 1 \) if \( \varphi_e(n) \uparrow \). Make clear in your argument where you use the fact that \( f \) is extensional.

[Hint: For each \( e, n \in \mathbb{N} \) consider the index \( i(e, n) \in \mathbb{N} \) of the register machine that inputs \( x \), computes \( \varphi_e(n) \) and if that computation halts, then computes \( \varphi_{e_1}(x) \).] [14 marks]

(ii) Deduce that if \( f \) is extensional, then \( S_f \) is either the whole of \( \mathbb{N} \), or not decidable. [3 marks]