2005 Paper 4 Question 6

Continuous Mathematics

(*a*) Let $f(x)$ be a periodic function of period 2π. Give expressions for the Fourier coefficients a_r $(r = 0, 1, \ldots)$ and b_r $(r = 1, 2, \ldots)$ of $f(x)$ where

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

is the Fourier series representation of $f(x)$. [2 marks]

(*b*) Show that the Fourier series in part (*a*) can also be written as a complex Fourier series

$$\sum_{r=-\infty}^{r=\infty} c_r e^{irx}$$

by deriving expressions for the complex Fourier coefficients c_r $(r = 0, \pm 1, \pm 2, \ldots)$ in terms of a_r and b_r. [3 marks]

(*c*) Use your expressions for a_r and b_r in part (*a*) and for c_r in part (*b*) to show that

$$c_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-irx} dx \qquad (r = 0, \pm 1, \pm 2, \ldots)\,.$$

[3 marks]

(*d*) Show that the complex Fourier coefficients of $f(x - \alpha)$ (where α is a constant) are given by $e^{-ir\alpha} c_r$ $(r = 0, \pm 1, \pm 2, \ldots)$. [6 marks]

(*e*) Suppose that $g(x)$ is another periodic function of period 2π with complex Fourier coefficients d_r $(r = 0, \pm 1, \pm 2, \ldots)$ and define $h(x)$ by

$$h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y) g(y) dy\,.$$

Show that $h(x)$ is a periodic function of period 2π and that its complex Fourier coefficients are given by $h_r = c_r d_r$ $(r = 0, \pm 1, \pm 2, \ldots)$. [6 marks]

[*You may assume that the periodic functions in this question satisfy the Dirichlet conditions. Euler's equation may be used without proof but should be stated precisely.*]