Continuous Mathematics

(a) Let \( f(x) \) be a periodic function of period \( 2\pi \). Give expressions for the Fourier coefficients \( a_r \) (\( r = 0, 1, \ldots \)) and \( b_r \) (\( r = 1, 2, \ldots \)) of \( f(x) \) where

\[
\frac{a_0}{2} + \sum_{r=1}^\infty (a_r \cos rx + b_r \sin rx)
\]

is the Fourier series representation of \( f(x) \). [2 marks]

(b) Show that the Fourier series in part (a) can also be written as a complex Fourier series

\[
\sum_{r=-\infty}^{\infty} c_re^{irx}
\]

by deriving expressions for the complex Fourier coefficients \( c_r \) (\( r = 0, \pm 1, \pm 2, \ldots \)) in terms of \( a_r \) and \( b_r \). [3 marks]

(c) Use your expressions for \( a_r \) and \( b_r \) in part (a) and for \( c_r \) in part (b) to show that

\[
c_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-irx} \, dx \quad (r = 0, \pm 1, \pm 2, \ldots)
\]

[3 marks]

(d) Show that the complex Fourier coefficients of \( f(x - \alpha) \) (where \( \alpha \) is a constant) are given by \( e^{-ir\alpha}c_r \) (\( r = 0, \pm 1, \pm 2, \ldots \)). [6 marks]

(e) Suppose that \( g(x) \) is another periodic function of period \( 2\pi \) with complex Fourier coefficients \( d_r \) (\( r = 0, \pm 1, \pm 2, \ldots \)) and define \( h(x) \) by

\[
h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y)g(y) \, dy.
\]

Show that \( h(x) \) is a periodic function of period \( 2\pi \) and that its complex Fourier coefficients are given by \( h_r = c_r d_r \) (\( r = 0, \pm 1, \pm 2, \ldots \)). [6 marks]

[You may assume that the periodic functions in this question satisfy the Dirichlet conditions. Euler’s equation may be used without proof but should be stated precisely.]