Computation Theory

(a) What does it mean for a subset $S$ of the set $\mathbb{N}$ of natural numbers to be register machine decidable? [3 marks]

(b) For each $e \in \mathbb{N}$, let $\varphi_e \in \text{Pfn}(\mathbb{N}, \mathbb{N})$ denote the partial function computed by the register machine program with index $e$. Let $e_0 \in \mathbb{N}$ be an index for the totally undefined partial function (so that $\varphi_{e_0}(x)^\uparrow$, for all $x \in \mathbb{N}$).

Suppose that a total function $f \in \text{Fun}(\mathbb{N}, \mathbb{N})$ is extensional, in the sense that for all $e, e' \in \mathbb{N}$, $f(e) = f(e')$ if $\varphi_e$ and $\varphi_{e'}$ are equal partial functions. Suppose also that the set $S_f = \{x \in \mathbb{N} \mid f(x) = f(e_0)\}$ is not the whole of $\mathbb{N}$, so that for some $e_1 \in \mathbb{N}$, $f(e_1) \neq f(e_0)$.

(i) If membership of $S_f$ were decided by a register machine $M$, show informally how to construct from $M$ a register machine $M'$ that, started with $R1 = e$ and $R2 = n$ (any $e, n \in \mathbb{N}$) always halts, with $R0 = 0$ if $\varphi_e(n)^\downarrow$ and with $R0 = 1$ if $\varphi_e(n)^\uparrow$. Make clear in your argument where you use the fact that $f$ is extensional.

[Hint: For each $e, n \in \mathbb{N}$ consider the index $i(e, n) \in \mathbb{N}$ of the register machine that inputs $x$, computes $\varphi_e(n)$ and if that computation halts, then computes $\varphi_{e_1}(x)$.] [14 marks]

(ii) Deduce that if $f$ is extensional, then $S_f$ is either the whole of $\mathbb{N}$, or not decidable. [3 marks]