Denotational Semantics

(a) The function \( \text{fix} \) is the least fixed point operator from \( (D \to D) \) to \( D \), for a domain \( D \).

(i) Show that \( \lambda f. f^n(\bot) \) is a continuous function from \( (D \to D) \) to \( D \) for any natural number \( n \).

[Hint: Use induction on \( n \). You may assume the evaluation function \( (f, d) \mapsto f(d) \) and the function \( f \mapsto (f, f) \), where \( f \in (D \to D) \) and \( d \in D \), are continuous.] [7 marks]

(ii) Now argue briefly why

\[
\text{fix} = \bigsqcup_{n \geq 0} \lambda f. f^n(\bot),
\]

to deduce that \( \text{fix} \) is itself a continuous function. [3 marks]

(b) In this part you are asked to consider a variant \( \text{PCF}_{\text{rec}} \) of the programming language \( \text{PCF} \) in which there are terms \( \text{rec} \, x : \tau. \, t \), recursively defining \( x \) to be \( t \), instead of terms \( \text{fix} \, \tau \).

(i) Write down a typing rule for \( \text{rec} \, x : \tau. \, t \). [2 marks]

(ii) Write down a rule for the evaluation of \( \text{rec} \, x : \tau. \, t \). [2 marks]

(iii) Write down the clause in the denotational semantics which describes the denotation of \( \text{rec} \, x : \tau. \, t \). (This will involve the denotation of \( t \) which you may assume.) [3 marks]

(iv) Write down a term in \( \text{PCF}_{\text{rec}} \) whose denotation is the least fixed point operator of type \( (\tau \to \tau) \to \tau \). [3 marks]