2004 Paper 8 Question 15

Denotational Semantics

(*a*) The function *fix* is the least fixed point operator from $(D \to D)$ to D, for a domain D.

(*i*) Show that $\lambda f.\ f^n(\bot)$ is a continuous function from $(D \to D)$ to D for any natural number n.
[Hint: Use induction on n. You may assume the evaluation function $(f, d) \mapsto f(d)$ and the function $f \mapsto (f, f)$, where $f \in (D \to D)$ and $d \in D$, are continuous.] [7 marks]

(*ii*) Now argue briefly why

$$\mathit{fix} = \bigsqcup_{n \geq 0} \lambda f.\ f^n(\bot)\ ,$$

to deduce that *fix* is itself a continuous function. [3 marks]

(*b*) In this part you are asked to consider a variant $\mathbf{PCF_{rec}}$ of the programming language **PCF** in which there are terms $\mathbf{rec}\, x : \tau.\, t$, recursively defining x to be t, instead of terms $\mathbf{fix}_\tau$.

(*i*) Write down a typing rule for $\mathbf{rec}\, x : \tau.\, t$. [2 marks]

(*ii*) Write down a rule for the evaluation of $\mathbf{rec}\, x : \tau.\, t$. [2 marks]

(*iii*) Write down the clause in the denotational semantics which describes the denotation of $\mathbf{rec}\, x : \tau.\, t$. (This will involve the denotation of t which you may assume.) [3 marks]

(*iv*) Write down a term in $\mathbf{PCF_{rec}}$ whose denotation is the least fixed point operator of type $(\tau \to \tau) \to \tau$. [3 marks]