Numerical Analysis II

(a) State a recurrence formula for the sequence of Chebyshev polynomials, \( \{T_k(x)\} \), and list these as far as \( T_5(x) \).

(b) What is the best \( L_\infty \) polynomial approximation over \([-1, 1]\) to \( x^k \) using polynomials of lower degree, and what is its degree? Use this property to explain the method of economisation of a Taylor series. How can the error in one economisation step be estimated?

(c) It is required to approximate the function \( f(x) = \lim_{k \to \infty} P_k(x) \) over \([-1, 1]\) with an absolute accuracy of 2 decimal places, where

\[
P_k(x) = \sum_{n=1}^{k} \frac{x^n}{n!}.
\]

As this series converges faster than \( e^x \), a good estimate of the error \( ||f(x) - P_k(x)||_\infty \) in the truncated Taylor series is given by evaluating the next term

\[
\frac{x^{k+1}}{(k+1)(k+1)!}
\]

at \( x = 1 \). Use the method of economisation to find a polynomial approximation of the required accuracy.