Logic and Proof

In this question \(x, y, z\) are variables, and \(a, b, c\) are constants.

(a) Briefly outline the semantics of first order logic. \[5\text{ marks}\]

(b) Use the semantics of first order logic to justify that the set of formulae

\[\{\forall x(x = c), \ P(a), \ \neg P(b)\}\]

is unsatisfiable. \[2\text{ marks}\]

(c) For each of the following first order logic formulae: either prove it to be valid using the sequent calculus; or give an interpretation that makes it false.

\[
\begin{align*}
\forall x(\exists y(R(x, y))) &\rightarrow \exists x(R(x, x)) \\
\exists x(\neg P(x)) &\rightarrow \neg \exists x(P(x)) \\
\neg \exists x(P(x)) &\rightarrow \exists x(\neg P(x)) \\
\exists x(P(x) \rightarrow P(a) \land P(b))
\end{align*}
\]

\[2\text{ marks each}\]

(d) Consider the following set \(\Gamma\) of first order logic formulae:

\[
\left\{ \begin{align*}
\forall x(\neg R(x, x)), \quad &\forall xyz(R(x, y) \land R(y, z) \rightarrow R(x, z)), \\
R(a, b), \quad &\forall xy(R(x, y) \rightarrow \exists z(R(x, z) \land R(z, y)))
\end{align*} \right\}
\]

(i) Find an interpretation that satisfies \(\Gamma\). \[3\text{ marks}\]

(ii) Can \(\Gamma\) be satisfied by an interpretation with a finite domain? Briefly justify your answer. \[2\text{ marks}\]