Numerical Analysis I

(a) The mid-point rule can be expressed in the form

\[ I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = f(n) + e_n \]

where

\[ e_n = f''(\theta_n)/24 \]

for some \( \theta_n \) in the interval \((n-\frac{1}{2}, n+\frac{1}{2})\). Assuming that a formula for \( \int f(x)dx \) is known, and using the notation

\[ S_{p,q} = \sum_{n=p}^{q} f(n), \]

describe a method for estimating the sum of a slowly convergent series \( S_{1,\infty} \), by summing only the first \( N \) terms and estimating the remainder by integration. [5 marks]

(b) Assuming that \( f''(x) \) is a positive decreasing function, derive an estimate of the error \( |E_N| \) in the method. [5 marks]

(c) Given

\[ \int \frac{dx}{(1 + x)\sqrt{x}} = 2\tan^{-1}\sqrt{x} \]

illustrate the method by applying it to

\[ \sum_{n=1}^{\infty} \frac{1}{(1 + n)\sqrt{n}}. \]

Verify that \( f''(x) \) is positive decreasing for large \( x \), and estimate the integral remainder to be added to \( S_{1,N} \). [6 marks]

(d) How large should \( N \) be to achieve an absolute error of approximately \( 2\times10^{-15} \)? [You may assume \( N + 1 \simeq N \) for this purpose.] [4 marks]