2003 Paper 10 Question 8

Mathematics for Computation Theory

Let A be a non-empty set. Define the *identity relation* Δ_A on A. [1 mark]

A *pre-order* on A is a relation R on A such that

(i) $\forall a \in A, (a,a) \in R$;

(ii) $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$.

Using a similar notation, specify additional conditions:

(iii), that must be satisfied in order that R be a partial order on A;

(iv), that in addition to (iii) must be satisfied in order that R be a total order on A. [2 marks]

Express conditions (i)–(iv) in terms of relations only (i.e. without reference to elements of A). [3 marks]

Suppose R is a pre-order on A. Let

$$S = \{(a,b) \mid (a,b) \in R \quad \textit{and} \quad (b,a) \in R\}.$$

Show that S is an equivalence relation on A. [4 marks]

Let $\frac{A}{S}$ be the set of S-equivalence classes. Write $[a]$ for $\{x \in A \mid (a,x) \in S\}$.

Define relation $\leqslant$ on $\frac{A}{S}$ as follows:

$$[a] \leqslant [b] \quad \text{iff} \quad (a,b) \in R.$$

Show that $\frac{A}{S}$ is partially ordered by $\leqslant$. [4 marks]

Let Z be the set of integers. Define the relation R on Z as follows:

$$\{(x,y) \in Z \times Z \mid \exists\, q \in Z \ s.t. \ y = xq\}.$$

Show that R is a pre-order on Z but *not* a partial order. Describe the derived partially ordered set $(\frac{Z}{S}, \leqslant)$. [4 marks]

What are the maximal and minimal elements in $(\frac{Z}{S}, \leqslant)$? [2 marks]