Let $A$ be a non-empty set. Define the identity relation $\Delta_A$ on $A$. [1 mark]

A **pre-order** on $A$ is a relation $R$ on $A$ such that

(i) $\forall a \in A, (a, a) \in R$;

(ii) $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

**Using a similar notation, specify additional conditions:**

(iii), that must be satisfied in order that $R$ be a partial order on $A$;

(iv), that in addition to (iii) must be satisfied in order that $R$ be a total order on $A$. [2 marks]

Express conditions (i)–(iv) in terms of relations only (i.e. without reference to elements of $A$). [3 marks]

Suppose $R$ is a pre-order on $A$. Let

$$S = \{(a, b) \mid (a, b) \in R \text{ and } (b, a) \in R\}.$$ 

Show that $S$ is an equivalence relation on $A$. [4 marks]

Let $\frac{A}{S}$ be the set of $S$-equivalence classes. Write $[a]$ for $\{x \in A \mid (a, x) \in S\}$.

Define relation $\leq$ on $\frac{A}{S}$ as follows:

$$[a] \leq [b] \text{ iff } (a, b) \in R.$$ 

Show that $\frac{A}{S}$ is partially ordered by $\leq$. [4 marks]

Let $Z$ be the set of integers. Define the relation $R$ on $Z$ as follows:

$$\{(x, y) \in Z \times Z \mid \exists q \in Z \text{ s.t. } y = xq\}.$$ 

Show that $R$ is a pre-order on $Z$ but not a partial order. Describe the derived partially ordered set $(\frac{Z}{S}, \leq)$. [4 marks]

What are the maximal and minimal elements in $(\frac{Z}{S}, \leq)$? [2 marks]