2002 Paper 6 Question 9

Semantics of Programming Languages

A call-by-value evaluation relation for λ-terms that are closed (i.e. ones without free variables) is inductively defined by the axiom

(1) $$\lambda x.\, M \Downarrow \lambda x.\, M$$

and the rule

(2) $$\frac{M_1 \Downarrow V_1 \quad M_2 \Downarrow V_2 \quad M[V_2/x] \Downarrow V}{(M_1\, M_2) \Downarrow V} \text{ if } V_1 = \lambda x.\, M$$

where V_1, V_2, V range over closed λ-abstractions and $M[V_2/x]$ denotes the result of substituting V_2 for all free occurrences of the variable x in the λ-term M. A call-by-value *applicative simulation* is a binary relation $\mathcal{S}$ between closed λ-terms satisfying that whenever $M_1\ \mathcal{S}\ M_2$ and $M_1 \Downarrow V_1$, then for some V_2 it is the case that $M_2 \Downarrow V_2$ and $(V_1\, V)\ \mathcal{S}\ (V_2\, V)$ for all V. Write $M_1 \leqslant M_2$ to mean that $M_1\ \mathcal{S}\ M_2$ holds for some such $\mathcal{S}$.

(a) Give a closed λ-term Ω with the property that $\Omega \Downarrow V$ holds for *no* V. [4 marks]

(b) Deduce that $\Omega \leqslant M$, for all M. [2 marks]

(c) Show that $M \leqslant M$, for all M. [2 marks]

(d) Show that $M[V/x] \leqslant (\lambda x.\, M)V$, for all M, V, x. [6 marks]

(e) Is it always the case that $M[N/x] \leqslant (\lambda x.\, M)N$ holds when N is not a λ-abstraction? [Hint: consider the case when $N = \Omega$ and M is a suitable λ-term not containing x free.] [6 marks]