Numerical Analysis II

(a) Taylor’s theorem states that if \( x \in [a, b] \) and \( f \in C^{N+1}[a, b] \)

\[
f(x) = T_N(a) + \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x - t)^N \, dt
\]

where

\[
T_N(a) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \cdots + \frac{(x - a)^N}{N!} f^{(N)}(a).
\]

Prove Taylor’s theorem. \[6\text{ marks}\]

(b) Peano’s theorem states that if a quadrature rule integrates polynomials of degree \( N \) exactly over an interval \([a, b]\) then the error in integrating \( f \in C^{N+1}[a, b] \) can be expressed as

\[
E(f) = \int_a^b f^{(N+1)}(t)K(t) \, dt
\]

where

\[
K(t) = \frac{1}{N!} E_x[(x - t)^N],
\]

Explain the notation \( E(f), E_x \) and \((x - t)^N\). \[4\text{ marks}\]

(c) Use Taylor’s theorem to prove Peano’s theorem. \[8\text{ marks}\]

(d) Under what additional condition may the simplified formula

\[
E(f) = \frac{f^{(N+1)}(\xi)}{(N + 1)!} E(x^{N+1})
\]

be applied? \[2\text{ marks}\]