(a) Define precisely what is meant by the following:

(i) \( f(x_1, x_2, \ldots, x_n) \) is a Primitive Recursive (PR) function of arity \( n \). [5 marks]

(ii) \( f(x_1, x_2, \ldots, x_n) \) is a Total Recursive (TR) function of arity \( n \). [3 marks]

(b) Ackermann’s function is defined by the following recursive scheme:

\[
\begin{align*}
f(0, y) &= S(y) = y + 1 \\
f(x + 1, 0) &= f(x, 1) \\
f(x + 1, y + 1) &= f(x, f(x + 1, y))
\end{align*}
\]

For fixed \( n \) define

\[ g_n(y) = f(n, y). \]

Show that for all \( n, y \in \mathbb{N} \),

\[ g_{n+1}(y) = g_{n}^{(y+1)}(1), \]

where \( h^{(k)}(z) \) is the result of \( k \) repeated applications of the function \( h \) to initial argument \( z \). [4 marks]

(c) Hence or otherwise show that for all \( n \in \mathbb{N}, g_n(y) \) is a PR function. [4 marks]

(d) Deduce that Ackermann’s function \( f(x, y) \) is a TR function. [3 marks]

(e) Is Ackermann’s function PR? [1 mark]