2001 Paper 3 Question 8

Continuous Mathematics

(*a*) The MacLaurin series for a continuous, infinitely differentiable function, $f(x)$, is:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

Derive the MacLaurin series for each of $\sin(x)$, $\cos(x)$, and e^x. [6 marks]

(*b*) Hence, or otherwise, prove that:

$$e^{i\phi} = \cos\phi + i\sin\phi$$

where $i = \sqrt{-1}$ [3 marks]

(*c*) Prove that the box function, $b(x)$:

$$b(x) = \begin{cases} 1, & |x| \leqslant \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

has the Fourier transform, $B(\nu)$:

$$B(\nu) = \frac{\sin\pi\nu}{\pi\nu}$$

where ν is frequency measured in Hertz (cycles per second). [7 marks]

(*d*) The convolution of $b(x)$ with itself is $t(x)$:

$$t(x) = b(x) * b(x) = \begin{cases} 1 + x, & -1 \leqslant x \leqslant 0 \\ 1 - x, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, or otherwise, find the Fourier transform, $T(\nu)$, of $t(x)$. [4 marks]