2001 Paper 2 Question 4

Probability

(a) Solve the following inhomogeneous difference equation:

$$u_n = 2(u_{n-1} + 3) \qquad \text{given that} \quad u_1 = 0$$

It may be assumed that $n \geqslant 1$. [5 marks]

A hardware device generates streams of ternary digits. Within a stream, each digit is equiprobably 0, 1 or 2. A stream ends as soon as each digit has been seen at least once. A stream may be as short as three digits (for example 201) but is usually rather longer (for example 1110102).

(b) Clearly there are three ways in which the first k digits of a stream may all be the same. What is the probability that the first k digits are all the same? [1 mark]

(c) By using the difference equation above, or otherwise, determine the number of ways in which the first k digits of a stream could comprise exactly two of the three available digits. [5 marks]

(d) What is the probability that the first k digits comprise exactly two of the three available digits? [1 mark]

(e) For $r \geqslant 2$, what is the probability that a stream is r digits long? [3 marks]

(f) What is the expected length of a stream? [5 marks]

Hint: It may be useful to note that

$$\sum_{r=1}^{\infty} r x^{r-1} = \frac{1}{(1-x)^2} \qquad \text{if} \quad 0 \leqslant x < 1$$