Discrete Mathematics

Let \((A, \leq_A)\) and \((B, \leq_B)\) be partially ordered sets.

(a) Define the product order on \(A \times B\) and prove that it is a partial order.  

The upper bound of a set \(S \subseteq A\) is an element \(u \in A\) (but not necessarily in \(S\)) such that \(\forall s \in S, s \leq u\). The least upper bound of \(S\) is an upper bound of \(S\) that is less than every other upper bound of \(S\). The greatest lower bound is defined similarly.

A lattice is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.

(b) Prove that \((\mathbb{N}, |)\), the natural numbers under the divisibility order, form a lattice.

(c) Given a set \(X\), prove that \((\mathcal{P}(X), \subseteq)\), the power set of \(X\) under set inclusion, forms a lattice.

(d) Does every subset of \((\mathbb{N}, |)\) have a least upper bound and a greatest lower bound? Justify your answer. What about \((\mathbb{N}_0, |)\) and \((\mathcal{P}(X), \subseteq)?\)

(e) If \((A, \leq_A)\) and \((B, \leq_B)\) are lattices, show that \(A \times B\) is a lattice under the product order.