Numerical Analysis II

(a) Taylor’s theorem states that if \( x \in [a,b] \) and \( f \in C^{N+1}[a,b] \)

\[
f(x) = T_N(a) + \frac{1}{N!} \int_a^x f^{(N+1)}(t)(x-t)^N \, dt
\]

where

\[
T_N(a) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^N}{N!} f^{(N)}(a).
\]

Prove Taylor’s theorem. [6 marks]

(b) Peano’s theorem states that if a quadrature rule integrates polynomials of degree \( N \) exactly over an interval \([a,b]\) then the error in integrating \( f \in C^{N+1}[a,b] \) can be expressed as

\[
E(f) = \int_a^b f^{(N+1)}(t)K(t) \, dt
\]

where

\[
K(t) = \frac{1}{N!} E_x[(x-t)^N].
\]

Explain the notation \( E(f) \), \( E_x \) and \( (x-t)^N \). [4 marks]

(c) Use Taylor’s theorem to prove Peano’s theorem. [8 marks]

(d) Under what additional condition may the simplified formula

\[
E(f) = \frac{f^{(N+1)}(\xi)}{(N+1)!} E(x^{N+1})
\]

be applied? [2 marks]