2001 Paper 11 Question 8

Computation Theory

(a) Define precisely what is meant by the following:

(i) $f(x_1, x_2, \dots x_n)$ is a Primitive Recursive (PR) function of arity n. [5 marks]

(ii) $f(x_1, x_2, \dots x_n)$ is a Total Recursive (TR) function of arity n. [3 marks]

(b) Ackermann's function is defined by the following recursive scheme:

$$\begin{aligned} f(0, y) &= S(y) = y + 1 \\ f(x+1, 0) &= f(x, 1) \\ f(x+1, y+1) &= f(x, f(x+1, y)) \end{aligned}$$

For fixed n define

$$g_n(y) = f(n, y).$$

Show that for all n, $y \in \mathbb{N}$,

$$g_{n+1}(y) = {g_n}^{(y+1)}(1),$$

where $h^{(k)}(z)$ is the result of k repeated applications of the function h to initial argument z. [4 marks]

(c) Hence or otherwise show that for all $n \in \mathbb{N}$, $g_n(y)$ is a PR function. [4 marks]

(d) Deduce that Ackermann's function $f(x, y)$ is a TR function. [3 marks]

(e) Is Ackermann's function PR? [1 mark]