Continuous Mathematics

(a) The MacLaurin series for a continuous, infinitely differentiable function, \( f(x) \), is:

\[
f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots
\]

Derive the MacLaurin series for each of \( \sin(x) \), \( \cos(x) \), and \( e^x \). [6 marks]

(b) Hence, or otherwise, prove that:

\[ e^{i\phi} = \cos \phi + i \sin \phi \]

where \( i = \sqrt{-1} \) [3 marks]

(c) Prove that the box function, \( b(x) \):

\[
b(x) = \begin{cases} 1, & |x| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}
\]

has the Fourier transform, \( B(\nu) \):

\[ B(\nu) = \frac{\sin \pi \nu}{\pi \nu} \]

where \( \nu \) is frequency measured in Hertz (cycles per second). [7 marks]

(d) The convolution of \( b(x) \) with itself is \( t(x) \):

\[
t(x) = b(x) * b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0 \\ 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}
\]

Hence, or otherwise, find the Fourier transform, \( T(\nu) \), of \( t(x) \). [4 marks]