2000 Paper 9 Question 13

Types

Give the axiom and rules of the type system for polymorphic lambda calculus (PLC). [6 marks]

Given any function ρ mapping type variables α to boolean values $\mathbf{b} \in \{\mathbf{true}, \mathbf{false}\}$, we extend ρ to a function on all PLC types by defining

$$\begin{aligned} \rho(\tau \to \tau') &= \rho(\tau) \Rightarrow \rho(\tau') \\ \rho(\forall\alpha(\tau)) &= \rho[\alpha \mapsto \mathbf{true}](\tau) \,\&\, \rho[\alpha \mapsto \mathbf{false}](\tau) \end{aligned}$$

where $\Rightarrow$ and $\&$ are the usual boolean operations of implication and conjunction, and where $\rho[\alpha \mapsto \mathbf{b}]$ is the function mapping α to $\mathbf{b}$ and otherwise acting like ρ. For example, show that for any ρ we have $\rho(\forall\alpha(\alpha \to \alpha)) = \mathbf{true}$ and $\rho(\forall\alpha(\alpha)) = \mathbf{false}$. [2 marks]

Let $\Phi(\Gamma, M, \tau)$ be the following property of PLC typing judgements $\Gamma \vdash M : \tau$.

> "For all ρ, if $\rho(\tau) = \mathbf{false}$ then Γ contains a type assignment $x_i : \tau_i$ with $\rho(\tau_i) = \mathbf{false}$."

Show that $\Phi(\Gamma, M, \tau)$ is closed under the axiom and rules of the PLC type system. (You may assume without proof that if α is not free in τ then $\rho[\alpha \mapsto \mathbf{b}](\tau) = \rho(\tau)$; and also that type substitutions $\tau'[\tau/\alpha]$ satisfy $\rho(\tau'[\tau/\alpha]) = \rho[\alpha \mapsto \rho(\tau)](\tau')$.) [10 marks]

Deduce that there is no *closed* PLC expression of type $\forall\alpha(\alpha)$. [2 marks]