Information Theory and Coding

(a) Prove that the information measure is additive: that the information gained from observing the combination of $N$ independent events, whose probabilities are $p_i$ for $i = 1 \ldots N$, is the sum of the information gained from observing each one of these events separately and in any order. [4 marks]

(b) What is the shortest possible code length, in bits per average symbol, that could be achieved for a six-letter alphabet whose symbols have the following probability distribution?

\[
\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32} \}
\]

[3 marks]

(c) Suppose that ravens are black with probability 0.6, that they are male with probability 0.5 and female with probability 0.5, but that male ravens are 3 times more likely to be black than are female ravens.

If you see a non-black raven, what is the probability that it is male? [4 marks]

How many bits worth of information are contained in a report that a non-black raven is male? [1 mark]

Rank-order for this problem, from greatest to least, the following uncertainties:

(i) uncertainty about colour
(ii) uncertainty about gender
(iii) uncertainty about colour, given only that a raven is male
(iv) uncertainty about gender, given only that a raven is non-black

[3 marks]

(d) If a continuous signal $f(t)$ is modulated by multiplying it with a complex exponential wave $\exp(i\omega t)$ whose frequency is $\omega$, what happens to the Fourier spectrum of the signal?

Name a very important practical application of this principle, and explain why modulation is a useful operation.

How can the original Fourier spectrum later be recovered? [3 marks]

(e) Which part of the 2D Fourier Transform of an image, the amplitude spectrum or the phase spectrum, is indispensable in order for the image to be intelligible?

Describe a demonstration that proves this. [2 marks]