Information Theory and Coding

(a) If a continuous signal is discretely sampled by multiplying it with a sequence of uniformly-spaced Dirac delta functions, having frequency $f_s$, what happens to the Fourier spectrum of the signal? [3 marks]

(b) What is the conditional probability $p(x|y)$, the probability of event $x$ given that event $y$ has occurred, provided that we know the following?

- $p(x)$, the unconditional probability of event $x$
- $p(y)$, the unconditional probability of event $y$
- $p(y|x)$, the probability of event $y$ given that event $x$ has occurred

[3 marks]

(c) Consider a binary symmetric communication channel, whose input source is the alphabet $X = \{0, 1\}$ with probabilities $\{0.5, 0.5\}$; whose output alphabet is $Y = \{0, 1\}$; and whose channel matrix is

\[
\begin{pmatrix}
1 - \epsilon & \epsilon \\
\epsilon & 1 - \epsilon
\end{pmatrix}
\]

where $\epsilon$ is the probability of transmission error.

(i) What is the entropy of the source, $H(X)$? [1 mark]

(ii) What is the probability distribution of the outputs, $p(Y)$, and the entropy of this output distribution, $H(Y)$? [3 marks]

(iii) What is the joint probability distribution for the source and the output, $p(X,Y)$, and what is the joint entropy, $H(X,Y)$? [3 marks]

(iv) What is the mutual information of this channel, $I(X;Y)$? [2 marks]

(v) How many values are there for $\epsilon$ for which the mutual information of this channel is maximal? What are those values, and what then is the capacity of such a channel in bits? [3 marks]

(vi) For what value of $\epsilon$ is the capacity of this channel minimal? What is the channel capacity in that case? [2 marks]