Numerical Analysis II

A Riemann integral over \([a, b]\) is defined by

\[
\int_a^b f(x) \, dx = \lim_{\Delta \xi \to 0} \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) f(x_i)
\]

Explain the terms Riemann sum and mesh norm. [4 marks]

With respect to an integral over \([-1, 1]\) which of the following are not Riemann sums? Give explanations.

(a) \(0.2f(-0.9) + 0.8f(-0.1) + 0.8f(+0.1) + 0.2f(+0.9)\)

(b) \(0.8f(-0.9) + 0.2f(-0.1) + 0.2f(+0.1) + 0.8f(+0.9)\)

(c) \(0.7f(-0.6) + 0.3f(-0.4) + 0.3f(+0.4) + 0.7f(+0.6)\)

(d) \(0.5f(-0.7) + 0.8f(0) + 0.5f(+0.7)\)

(e) \(0.3f(-0.7) + 1.0f(+0.1) + 0.7f(+0.7)\) [5 marks]

Suppose \(\mathbf{R}\) is a rule that integrates constants exactly over \([-1, 1]\), and \(f(x)\) is bounded and Riemann-integrable over \([a, b]\). Write down a formula for the composite rule \((n \times \mathbf{R})f\) and prove that

\[
\lim_{n \to \infty} (n \times \mathbf{R})f = \int_a^b f(x) \, dx
\] [6 marks]

Which of the examples (a) to (e) converge in composite form? [2 marks]

Does the rule \(-0.5f(-1) + 1.5f(-0.4) + 1.5f(+0.4) - 0.5f(+1)\) converge in composite form? Comment on its suitability for this purpose. [3 marks]