Types

The terms of the untyped lambda calculus, $M ::= x \mid \lambda x(M) \mid MM$, are to be assigned types of the form $\tau ::= \alpha \mid \tau \to \tau$, where $\alpha$ ranges over an infinite set of type variables. Give an inductive definition of a typing judgement of the form $A, \Delta \vdash M : \tau$, where $\Delta$ is a finite function from variables to types whose domain of definition contains the free variables of $M$, and where $A$ is a finite set of type variables containing the type variables occurring in $\tau$ and $\Delta$. [3 marks]

Write $\text{Typ}(A)$ for the set of types involving only type variables in the set $A$. Let $A, A', A''$ be finite sets of type variables; $S$ be a function from $A$ to $\text{Typ}(A')$ and $T$ a function from $A'$ to $\text{Typ}(A'')$; $\tau_1, \tau_2$ be types in $\text{Typ}(A)$; and $\tau'$ be a type in $\text{Typ}(A')$. Give definitions of the following concepts:

(a) The type $S(\tau_1)$ resulting from simultaneously substituting the type $S(\alpha)$ for occurrences of $\alpha$ in $\tau_1$, as $\alpha$ ranges over $A$. [2 marks]

(b) The composition $TS : A \to \text{Typ}(A'')$ of the type substitutions $S$ and $T$. [2 marks]

(c) $S$ unifies $\tau_1$ and $\tau_2$. [2 marks]

(d) $S$ is the most general unifier of $\tau_1$ and $\tau_2$. [2 marks]

(e) $(S, \tau')$ is a typing for a partial typing judgement $A, \Delta \vdash M : ?$. [2 marks]

(f) $(S, \tau')$ is a principal typing for a partial typing judgement $A, \Delta \vdash M : ?$. [2 marks]

Give examples, with proof, of closed lambda terms $M_1$ and $M_2$ for which $\emptyset, \emptyset \vdash M_1 : ?$ has a typing and $\emptyset, \emptyset \vdash M_2 : ?$ does not. [4 marks]

If a partial typing judgement has a typing, does it necessarily have a principal one? [1 mark]