Continuous Mathematics

(a) In his formulation of the calculus, Newton captured only the notion of integer-order differentiation considering first-, second- and third-order derivatives, and so on. In scientific computing, however, we sometimes need fractional-order derivatives, as for example in fluid mechanics.

Explain how Fractional Differentiation (derivatives of non-integer order) can be given precise quantitative meaning through Fourier analysis. [5 marks]

Suppose that a continuous function \( f(x) \) has Fourier Transform \( F(\mu) \). Outline an algorithm (as a sequence of mathematical steps, not an actual program) for computing the 1.5\(^{th}\) derivative of some function \( f(x) \)

\[
\frac{d^{(1.5)} f(x)}{dx^{(1.5)}}
\]  

[5 marks]

(b) For \( i = \sqrt{-1} \), consider the quantity \( \sqrt{i} \).

(i) Express \( \sqrt{i} \) as a complex exponential. [1 mark]

(ii) In which quadrant of the complex plane does it lie? [1 mark]

(iii) What is the real part of \( \sqrt{i} \)? [1 mark]

(iv) What is the imaginary part of \( \sqrt{i} \)? [1 mark]

(v) What is the length (the modulus) of \( \sqrt{i} \)? [1 mark]

(c) Initial-value problems described by ordinary differential equations have solutions that can be propagated forward using incrementing rules such as Euler or Runge–Kutta. But boundary-value problems specified by partial differential equations (PDEs) such as Poisson’s Equation,

\[
\frac{\partial^2 \mu(x,y)}{\partial x^2} + \frac{\partial^2 \mu(x,y)}{\partial y^2} = \rho(x,y)
\]

cannot be solved by such propagation methods. Why not? [3 marks]

State the principle for one general class of numerical methods for solving such PDEs. [2 marks]