1999 Paper 8 Question 14

Numerical Analysis II

State a recurrence formula for the sequence of Chebyshev polynomials, $\{T_n(x)\}$, and list these as far as $T_5(x)$. [4 marks]

What is the best polynomial approximation over $[-1, 1]$ to x^n using polynomials of lower degree, and what is its degree? Use this property to explain the method of economisation of a Taylor series. How can the error in one economisation step be estimated? [7 marks]

The error in Lagrange interpolation can be expressed in the form

$$f(x) - L_{n-1}(x) = \frac{f^n(\xi)}{n!} \prod_{j=1}^{n} (x - x_j)$$

for a suitable function $f(x)$. What is the best choice for abscissae $\{x_j\}$ and why? [2 marks]

The function $\sin x$ may be approximated by the truncated Taylor series

$$P_{2n-1}(x) = \sum_{i=1}^{n} (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!}.$$

Estimate the maximum absolute error over $[-1, 1]$ for both $P_3(x)$ and $P_5(x)$. Perform one economisation step on $P_5(x)$ and show that the resulting polynomial is more accurate than $P_3(x)$. [7 marks]