Numerical Analysis I

The mid-point rule can be expressed in the form

\[ I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = f(n) + e_n \]

where

\[ e_n = f''(\theta_n)/24 \]

for some \( \theta_n \) in the interval \((n - \frac{1}{2}, n + \frac{1}{2})\). Assuming that a formula for \( \int f(x)dx \) is known, and using the notation

\[ S_{p,q} = \sum_{n=p}^{q} f(n), \]

describe a method for estimating the sum of a slowly convergent series \( S_{1,\infty} \), by summing only the first \( N \) terms and estimating the remainder by integration. [6 marks]

Assuming that \( f''(x) \) is a positive decreasing function, derive an estimate of the error \( |E_N| \) in the method. [5 marks]

Given

\[ \int \frac{dx}{x(x+2)} = -\frac{1}{2} \log_e(1 + \frac{2}{x}) \]

illustrate the method by applying it to

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+2)}. \]

Verify that \( f''(x) \) is positive decreasing for large \( x \), and estimate the integral remainder to be added to \( S_{1,N} \). [You may assume \( \log_e(1 + \lambda) \simeq \lambda \) for \( \lambda \) small.] [6 marks]

To 2 significant digits, how large should \( N \) be to achieve an absolute error of approximately \( 1.8 \times 10^{-11} \)? [3 marks]