**Numerical Analysis II**

State a recurrence formula for the sequence of Chebyshev polynomials, \( \{T_n(x)\} \), and list these as far as \( T_5(x) \). [4 marks]

What is the best polynomial approximation over \([-1, 1]\) to \( x^n \) using polynomials of lower degree, and what is its degree? Use this property to explain the method of economisation of a Taylor series. How can the error in one economisation step be estimated? [7 marks]

The error in Lagrange interpolation can be expressed in the form

\[
f(x) - L_{n-1}(x) = \frac{f^n(\xi)}{n!} \prod_{j=1}^{n} (x - x_j)
\]

for a suitable function \( f(x) \). What is the best choice for abscissae \( \{x_j\} \) and why? [2 marks]

The function \( \sin x \) may be approximated by the truncated Taylor series

\[
P_{2n-1}(x) = \sum_{i=1}^{n} (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!}.
\]

Estimate the maximum absolute error over \([-1, 1]\) for both \( P_3(x) \) and \( P_5(x) \). Perform one economisation step on \( P_5(x) \) and show that the resulting polynomial is more accurate than \( P_3(x) \). [7 marks]