1999 Paper 11 Question 13

Numerical Analysis I

The mid-point rule can be expressed in the form

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = f(n) + e_n$$

where

$$e_n = f''(\theta_n)/24$$

for some θ_n in the interval $(n - \frac{1}{2}, n + \frac{1}{2})$. Assuming that a formula for $\int f(x)dx$ is known, and using the notation

$$S_{p,q} = \sum_{n=p}^{q} f(n),$$

describe a method for estimating the sum of a slowly convergent series $S_{1,\infty}$, by summing only the first N terms and estimating the remainder by integration.

[6 marks]

Assuming that $f''(x)$ is a positive decreasing function, derive an estimate of the error $|E_N|$ in the method. [5 marks]

Given

$$\int \frac{dx}{x(x+2)} = -\frac{1}{2} \log_e(1 + \frac{2}{x})$$

illustrate the method by applying it to

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

Verify that $f''(x)$ is positive decreasing for large x, and estimate the integral remainder to be added to $S_{1,N}$. [You may assume $\log_e(1 + \lambda) \simeq \lambda$ for λ small.]

[6 marks]

To 2 significant digits, how large should N be to achieve an absolute error of approximately 1.8×10^{-11}? [3 marks]