Continuous Mathematics

Many important problems in mathematical modelling and scientific computing require the use of complex variables. Unfortunately, popular programming languages like C do not have a complex variable type, and so we must construct them from floating-point types. Assuming that the quantities $a, b, c, d$ are all real numbers and $i = \sqrt{-1}$, resolve the following expressions, or explain the following operations, involving complex variables $Z_1 = a + ib$ and $Z_2 = c + id$:

(a) Let $Z_3 = Z_1 Z_2$. What is the real part of $Z_3$, and what is its imaginary part? [2 marks]

(b) What is $\|Z_1\|$, the modulus of $Z_1$, and what is $\|Z_3\|$, the modulus of $Z_3 = Z_1 Z_2$? [2 marks]

(c) What is $\angle Z_2$, the angle of complex variable $Z_2$? [2 marks]

(d) Express $Z_1$ in complex polar form, not using the quantities $a$ or $b$ but rather the modulus $\|Z_1\|$ and angle $\angle Z_1$. [2 marks]

(e) Suppose that $Z_1$ and $Z_2$ both have a modulus of 1. Explain, with the aid of a diagram, how their product $Z_3 = Z_1 Z_2$ amounts to a rotation in the complex plane. Why is the multiplication of these complex variables reduced now to addition? Without using the quantities $a, b, c, d$, what is the value of $\|Z_3\|$? [4 marks]

(f) Suppose that in complex polar form, $Z = \exp(2\pi i/5)$. What do you get if $Z$ is multiplied by itself 5 times? Give the simplest possible answer that you can. [2 marks]

(g) Consider the complex exponential function $f(x) = \exp(2\pi i \omega x)$. What function is its real part? What function is its imaginary part? [2 marks]

(h) If the above function $f(x)$ passes through a linear system, i.e. is operated upon by any conceivable linear differential or integral operator, what is the most dramatic way in which $f(x)$ can possibly be affected? [4 marks]