1998 Paper 7 Question 5

Denotational Semantics

Suppose that D is a domain and that $lam : (D \to D) \to D$ and $app : D \to (D \to D)$ are continuous functions. Using D, lam and app, you are required to give a denotational semantics to the terms of the untyped lambda calculus: $M ::= x \mid \lambda\, x\, (M) \mid M\, M$, where x ranges over some fixed, infinite set of variables and where terms are identified up to alpha-conversion. For each term M and list $\vec{x} = x_1, \ldots, x_n$ of distinct variables containing the free variables of M, define a continuous function

$$\rho \mapsto [\![\vec{x} \vdash M]\!](\rho)$$

mapping elements ρ of the product domain D^n (regarded as functions from $\{x_1, \ldots, x_n\}$ to D) to elements of D. The definition should proceed by induction on the structure of M and you should state clearly, but without proof, any properties of continuous functions between domains which are needed for the definition to make sense. [10 marks]

Show, by induction on the structure of M, that the following substitution property holds:

$$[\![\vec{x} \vdash M[M'/x]]\!](\rho) = [\![\vec{x}, x \vdash M]\!](\rho[x \mapsto [\![\vec{x} \vdash M']\!](\rho)]).$$

(You may assume without proof that $[\![\vec{x}, x \vdash M]\!](\rho[x \mapsto d]) = [\![\vec{x} \vdash M]\!](\rho)$ when x does not occur free in M.) [5 marks]

Show that if the composition $app \circ lam$ is the identity function on the function domain $D \to D$, then the denotational semantics respects beta-reduction, in the sense that $[\![\vec{x} \vdash (\lambda\, x\, (M))\, M']\!](\rho) = [\![\vec{x} \vdash M[M'/x]]\!](\rho)$. [3 marks]

What condition on lam and app will ensure that eta-reduction, $\lambda\, x\, (M x) \to M$ (where x is not free in M), is respected? [2 marks]