Types

Each natural number \( n \in \mathbb{N} \) can be encoded in the polymorphic lambda calculus (PLC) by the beta-normal form \( \text{Num}_n \overset{\text{def}}{=} \Lambda \alpha (\lambda x : \alpha (\lambda f : \alpha \rightarrow \alpha (f^n x))) \), where the expression \( f^n x \) is an abbreviation for the PLC expression inductively defined by
\[
f^0 x \overset{\text{def}}{=} x \quad \text{and} \quad f^{n+1} x \overset{\text{def}}{=} f (f^n x).
\]

For which PLC type \( \text{nat} \) does \( \vdash \text{Num}_n : \text{nat} \) hold? \[3 \text{ marks}\]

Say that a function \( \phi : \mathbb{N} \rightarrow \mathbb{N} \) is PLC-representable if there is a PLC expression \( F \) such that the following typing and beta-conversions hold:
\[
\vdash F : \text{nat} \rightarrow \text{nat} \quad \text{and} \quad F \text{Num}_n =_\beta \text{Num}_{\phi(n)} \quad (\text{all } n \in \mathbb{N}).
\]

Show that the successor function, \( s(n) \overset{\text{def}}{=} n + 1 \), is PLC-representable. \[5 \text{ marks}\]

Given a PLC-representable function \( \phi \) and a number \( a \in \mathbb{N} \), show that the function \( \psi \) inductively defined by
\[
\psi(0) \overset{\text{def}}{=} a \quad \text{and} \quad \psi(n + 1) \overset{\text{def}}{=} \phi(\psi(n))
\]
is PLC-representable. \[8 \text{ marks}\]

Is every function \( \mathbb{N} \rightarrow \mathbb{N} \) PLC-representable? Justify your answer. \[4 \text{ marks}\]