

1998 Paper 5 Question 12

Semantics of Programming Languages

An abstract machine for evaluating closed terms of the untyped lambda calculus has configurations which are non-empty lists of closed terms. Its transitions are of two forms:

$$\begin{array}{l} (\overrightarrow{\text{app}}) \quad (M_1 M_2) :: L \rightarrow M_1 :: M_2 :: L \\ (\overrightarrow{\text{abs}}) \quad \lambda x (M_1) :: M_2 :: L \rightarrow M_1[M_2/x] :: L \end{array}$$

where $::$ denotes list concatenation and $M_1[M_2/x]$ denotes the result of substituting M_2 for all free occurrences of the variable x in M_1 . Let \Downarrow be the binary relation between closed terms inductively defined by the following axioms and rules:

$$\begin{array}{l} (\Downarrow_{\text{abs}}) \quad \lambda x (M) \Downarrow \lambda x (M) \\ (\Downarrow_{\text{app}}) \quad \frac{M_1 \Downarrow \lambda x (M_2) \quad M_2[M_3/x] \Downarrow \lambda x (M_4)}{M_1 M_3 \Downarrow \lambda x (M_4)}. \end{array}$$

- (a) Prove by Rule Induction that if $M_1 \Downarrow \lambda x (M_2)$ holds, then so does $M_1 :: L \rightarrow^* \lambda x M_2 :: L$, where \rightarrow^* denotes the reflexive-transitive closure of the transition relation \rightarrow . [5 marks]
- (b) Prove by Mathematical Induction on n that if $(\dots((M[M_0/x] M_1)M_2)\dots)M_n \Downarrow \lambda x (M')$, then $(\dots(((\lambda x (M))M_0)M_1)M_2)\dots)M_n \Downarrow \lambda x (M')$. [5 marks]
- (c) Given a configuration $M :: L$, let $M@L$ denote the closed term defined by induction on the length of the list L by:
 $M@\mathbf{nil} \stackrel{\text{def}}{=} M$ and $M@(M' :: L) \stackrel{\text{def}}{=} (M M')@L$. Using (b), show by case analysis for \rightarrow that if $M_1 :: L_1 \rightarrow M_2 :: L_2$ and $M_2@L_2 \Downarrow \lambda x (M')$ hold, then so does $M_1@L_1 \Downarrow \lambda x (M')$. [5 marks]
- (d) Deduce from (a) and (c) that $M_1 \Downarrow \lambda x (M_2)$ holds if and only if $M_1 :: \mathbf{nil} \rightarrow^* \lambda x (M_2) :: \mathbf{nil}$ does. [5 marks]