Define the Chebyshev polynomial \( T_k(x) \). Evaluate \( T_4\left(\frac{1}{2}\right) \) using the formula \( T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \). What is the leading coefficient of \( T_k(x) \)? [4 marks]

The best \( L_\infty \) approximation to \( f(x) \in C[-1,1] \) by a polynomial \( p_{n-1}(x) \) of degree \( n-1 \) has the property that

\[
\max_{x \in [-1,1]} |e(x)|
\]

is attained at \( n + 1 \) distinct points \(-1 \leq \xi_0 < \xi_1 < \ldots < \xi_n \leq 1\) such that \( e(\xi_j) = -e(\xi_{j-1}) \) for \( j = 1, 2, \ldots n \).

Let \( f(x) = x^2 \). Show, by means of a clearly labelled sketch graph, that the best polynomial approximation of degree 1 is a constant. [3 marks]

Now suppose \( f(x) = x^3 \) is the function to be approximated. Taking account of symmetry, sketch the graph of \( f(x) \) and its best \( L_\infty \) approximation by a polynomial of degree 2. [5 marks]

By differentiating \( e(x) \), find the polynomial \( p_2(x) \). [6 marks]

State a formula for the best approximation to \( f(x) = x^n \) by a polynomial of degree \( n-1 \). [2 marks]