

## 1996 Paper 6 Question 12

### Semantics

If  $D$  is a complete partial order and  $C \subseteq D$ , then we say that  $C$  is a *closed* subset of  $D$  if it satisfies the following two conditions:

- For all  $x \in D$  and  $y \in C$ , if  $x \sqsubseteq y$  then  $x \in C$ .
- If  $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \dots$  is an  $\omega$ -chain such that  $x_n \in C$  for all  $n$ , then  $(\bigsqcup_{n \in \omega} x_n) \in C$ .

Prove the following statements:

- (a) If  $C_i$  is a closed subset of  $D$  for all  $i \in I$ , then  $\bigcap_{i \in I} C_i$  is a closed subset of  $D$ . [4 marks]
- (b) If  $C_1, C_2, \dots, C_N$  are closed subsets of  $D$ , then  $\bigcup_{1 \leq i \leq N} C_i$  is a closed subset of  $D$ . [7 marks]
- (c) If  $x \in D$ , then  $\downarrow x \stackrel{\text{def}}{=} \{y \in D \mid y \sqsubseteq x\}$  is a closed subset of  $D$ . [4 marks]
- (d) If  $f : E \rightarrow D$  is a continuous function, then for all closed subsets  $C$  of  $D$ ,  $f^{-1}(C)$  is a closed subset of  $E$ . (Where  $f^{-1}(C) \stackrel{\text{def}}{=} \{x \in E \mid f(x) \in C\}$ .) [5 marks]