1996 Paper 10 Question 9

Mathematics for Computation Theory

Let S be a finite alphabet, $\mathcal{E}$ be an algebra of events over S, and M_{mn} be the algebra of $(m \times n)$ event matrices over $\mathcal{E}$. For M, $N \in M_{mn}$ define

$$M \leqslant N \quad \text{if and only if} \quad M + N = N$$

so that $(M, \leqslant)$ is a partially ordered set.

Let A, B be $(m \times m)$, $(m \times n)$ event matrices over $\mathcal{E}$. Prove that $X = A^*B$ is the least $(m \times n)$ event matrix such that

$$X = B + AX \tag{$*$}$$

stating clearly any algebraic assumptions on which your proof depends (Arden's rule). [10 marks]

Suppose now that $A = (A_{ij} \mid 1 \leqslant i, j \leqslant m)$ is such that no event A_{ij} contains the null string. Prove that $X = A^*B$ is the *unique* solution of $(*)$.

[Hint: suppose if possible $X > A^*B$ is a solution of $(*)$. Let $(X - A^*B) = Y = (Y_{ij})$, and choose string y of minimal length across all Y_{ij}.] [10 marks]